Periodic potentials and supersymmetry

Periodic potentials and supersymmetry

Avinash Khare[1] and Uday Sukhatme[2]

[1] Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, India
[2] Department of Physics, State University of New York at Buffalo, Buffalo, NY 14260, USA

E-mail: khare@iopb.res.in and sukhatme@buffalo.edu

Abstract
We review the current status of one-dimensional periodic potentials and
also present several new results. It is shown that using the formalism of
supersymmetric quantum mechanics, one can considerably enlarge the limited
class of analytically solvable one-dimensional periodic potentials. Further,
using the Landen transformations as well as cyclic identities for Jacobi elliptic
functions discovered by us recently, it is shown that a linear superposition of
Lamé (as well as associated Lamé) potentials is also analytically solvable.
Finally, using anti-isospectral transformations, we also obtain a class of
analytically solvable, complex, PT-invariant, periodic potentials having real
band spectra.

PACS number: 03.65.−w

1. Introduction

One-dimensional potential wells have bound states. They are solutions of the Schrödinger
equation which satisfy appropriate boundary conditions. When two identical potential wells
are very far apart, then each potential has the same energy levels, and each eigenstate is doubly
degenerate. As the wells are brought closer together, there is communication between them,
and each level is split into two. Similarly, if one has an array of many identical wells making
a periodic potential, then one gets energy bands, a concept which is well explained in many
quantum mechanics books [1]. Energy bands play a crucial role in determining the electronic
properties of crystalline solids. To illustrate this band structure qualitatively, condensed matter
physics texts usually treat the problem of a one-dimensional periodic array of delta functions,
called the Kronig–Penney model. Here, one gets a transcendental equation for computing

band edges. Another well-studied class of periodic potentials is the Lamé potentials [2–4]

$$V(x) = a(a+1)m \, \text{sn}^2(x, m) \tag{1}$$

where $\text{sn}(x, m)$ is a Jacobi elliptic function [5][3] of real elliptic modulus parameter m $(0 \leqslant m \leqslant 1)$. If a is any positive integer, it is well known that these potentials have only a band gaps and hence $2a + 1$ band edges which are analytically known. However, this is an exceptional example. The bottom line is that even in one dimension, there are very few solvable periodic potentials, and it would be desirable to have more, especially some with a richer spatial structure.

We shall describe how to obtain new solvable periodic potentials via several different approaches. One way, for example, is to expand our knowledge of Lamé potentials to the wider class of associated Lamé (AL) potentials [6, 7]

$$V(x) = a(a+1)m \, \text{sn}^2(x, m) + b(b+1)m \frac{\text{cn}^2(x, m)}{\text{dn}^2(x, m)} \tag{2}$$

where a, b are constants and without any loss of generality we consider $a \geqslant b$. A second way, for example, is to exploit the fact that Jacobi elliptic functions are doubly periodic functions and consider PT-invariant complex periodic potentials [8] obtained from the Lamé and the AL potentials by considering the anti-isospectral transformation $x \to \text{i}x + \beta$ (where β is any nonzero real number) [9]. A third way is to consider a linear superposition of several Lamé (AL) potentials which in view of the Landen transformations [10, 11] and cyclic identities [12] for Jacobi elliptic functions can be shown to be essentially equivalent to Lamé (AL) potentials with modified parameters. Finally, we can further expand all these classes of solvable potentials by using the techniques of supersymmetric quantum mechanics to generate supersymmetric partner potentials [13].

The outline of this paper is as follows. In section 2, we review some general properties of periodic potentials as well as the relevant formalism of supersymmetric quantum mechanics. In section 3, we discuss the band edge energies and eigenfunctions for several Lamé and AL potentials. In particular, for Lamé potentials (1) we derive a remarkable relation connecting band edge energy eigenstates corresponding to modulus parameters m and $1 - m$. We also describe how the Lamé potential results can be vastly expanded to get solutions for the AL potentials (2). In particular, we show that for any integral a, b with $a > b$, there are a bound bands followed by a continuum band, out of which the top b bound bands are unusual in the sense that both band edges are of period $2K(m)$ $(4K(m))$ when the integer $a - b$ is even (odd), where $K(m)$ is the complete elliptic integral of the first kind. In section 4, we obtain the supersymmetric partners of both the Lamé and the AL potentials, thereby expanding the list of solvable periodic potentials. In section 5, we show that using Landen transformations [11] and cyclic identities obtained by us recently [12], exact band edge eigenvalues and eigenfunctions can also be obtained in the case of linear superposition of Lamé and AL potentials. In section 6, we show that a class of PT-invariant complex periodic potentials obtained from the Lamé and AL potentials by the anti-isospectral transformation $x \to \text{i}x + \beta$ (β being any nonzero real number) are also exactly solvable problems with real band spectra. Finally, in section 7, we discuss the double sine-Gordon equation and show that in some special cases it has unusual band spectra, in that the majority of bound bands have both their band edges of the same period.

---

[3] The modulus parameter m is often called k^2 in the mathematics literature. The related complementary quantity $(1 - m)$ is often called k'^2.

2. General properties

2.1. Periodic potentials

For a real potential with period L, one is seeking solutions of the Schrödinger equation subject to the Bloch condition

$$\psi(x) = e^{ikL}\psi(x+L) \tag{3}$$

where k denotes the crystal momentum. The spectrum shows energy bands whose edges correspond to $kL = 0, \pi$, that is the wavefunctions at the band edges satisfy $\psi(x) = \pm\psi(x+L)$. For periodic potentials, the band edge energies and wavefunctions are often called eigenvalues and eigenfunctions, and we will also use this terminology.

A general property of eigenstates for a real potential with period L is the oscillation theorem. It states that band edge wavefunctions, when arranged in order of increasing energy $E_0 < E_1 \leqslant E_2 < E_3 \leqslant E_4 < E_5 \leqslant E_6 < \cdots$, have periods $L, 2L, 2L, L, L,$ $2L, 2L, \ldots$ [2]. The corresponding number of wavefunction nodes in the interval L are $0, 1, 1, 2, 2, 3, 3, \ldots$ and the energy band gaps are given by $\Delta_1 \equiv E_2 - E_1, \Delta_2 \equiv E_4 - E_3, \Delta_3 \equiv E_6 - E_5, \ldots$. Note that it is possible for any band gap Δ to vanish, in which case the two adjacent bands merge into one. The oscillation theorem is very useful in identifying if all band edge eigenstates have been properly determined or if some have been missed.

2.2. Supersymmetric quantum mechanics

The supersymmetric partner potentials $V_\pm(x)$ are defined in terms of the superpotential $W(x)$ by [13]

$$V_\pm(x) = W^2(x) \pm W'(x) \tag{4}$$

where (by convention) the superpotential $W(x)$ is related to the ground state eigenfunction of $V_-(x)$ by

$$W(x) = -\frac{\psi_0'^{(-)}(x)}{\psi_0^{(-)}(x)}. \tag{5}$$

The corresponding Hamiltonians $H_\pm$ can be factorized as

$$H_- = A^+ A \qquad H_+ = AA^+ \tag{6}$$

where

$$A = \frac{d}{dx} + W(x) \qquad A^+ = -\frac{d}{dx} + W(x) \tag{7}$$

so that the spectra of $H_\pm$ are nonnegative. It is also clear that on the full line, both $H_\pm$ cannot have zero energy modes, since the two functions $\psi_0^{(\pm)}$ given by

$$\psi_0^{(\pm)}(x) = \exp\left(\pm\int^x W(y)\,dy\right) \tag{8}$$

cannot be simultaneously normalized.

On the other hand, for an arbitrary choice of the superpotential $W(x)$ with period L [$W(x+L) = W(x)$], the eigenfunctions of both $V_\pm(x)$ must satisfy the Bloch condition (3). In this periodic situation, irrespective of whether supersymmetry is broken or unbroken,

the spectra of $V_{\pm}(x)$ are strictly identical [14, 15]. Further, one has unbroken supersymmetry provided

$$\int_0^L W(y)\, dy = 0 \tag{9}$$

and in this case the partner Hamiltonians have identical spectra including zero modes and both $\psi_0^{(\pm)}$ belong to the Hilbert space. As a result, unlike on the full line, for periodic potentials, whenever condition (9) is satisfied, supersymmetry is unbroken and yet the Witten index (which counts the difference between the number of zero modes of $\psi_0^{(\pm)}$) is always zero.

The condition (9) is trivially satisfied when $W(x)$ is an odd function of x and throughout this paper we shall only consider superpotentials W which are odd functions of x. Further, using the known eigenfunctions $\psi_n^{(-)}(x)$ of $V_-(x)$, one can immediately write the corresponding eigenfunctions $\psi_n^{(+)}(x)$ of $V_+(x)$. In particular, from equation (8) it follows that the ground state of $V_+(x)$ is given by [13]

$$\psi_0^{(+)}(x) = \frac{1}{\psi_0^{(-)}(x)} \tag{10}$$

while the un-normalized excited states $\psi_n^{(+)}(x)$ are obtained from $\psi_n^{(-)}(x)$ by using the relation

$$\psi_n^{(+)}(x) = \left[\frac{d}{dx} + W(x)\right] \psi_n^{(-)}(x) \qquad (n \geqslant 1). \tag{11}$$

Thus by starting from an exactly solvable periodic potential $V_-(x)$, one gets a new isospectral periodic potential $V_+(x)$.

A few years ago, the concept of self-isospectral periodic potentials was defined and developed in detail [14]. A one-dimensional potential $V_-(x)$ of period L is said to be self-isospectral if its supersymmetric partner potential $V_+(x)$ is just the original potential up to a discrete transformation—a translation by any constant amount, a reflection or both. A common example is translation by half a period, in which case the condition for self-isospectrality is $V_+(x) = V_-(x - L/2)$. In this sense, any self-isospectral potential is rather uninteresting, since the application of supersymmetry just yields a discrete transformation and basically nothing new. For completeness, we mention here that the above discussion of first-order supersymmetric transformations can be profitably extended to second-order transformations, which results in additional, physically acceptable, solvable, periodic potentials [16], which are sometimes self-isospectral [17].

3. Lamé and associated Lamé potentials

3.1. Lamé potentials

The Lamé potentials as given by equation (1) have a period $L = 2K(m)$. The name is derived from the fact that the corresponding Schrödinger equation (with $\hbar = 2m = 1$)

$$-\frac{d^2\psi}{dx^2} + a(a+1)m\, \text{sn}^2(x, m)\psi = E\psi \tag{12}$$

is called the Lamé equation [2, 3]. Lamé potentials have been treated in classical mechanics contexts by many people [18]. In a quantum mechanical context, it is well known that for any integer value $a = 1, 2, 3, \ldots$, the corresponding Lamé potential has a bound bands followed by a continuum band [2, 3] and the $2a+1$ band edge energy eigenstates are analytically known. We now obtain remarkable relations (valid for any integer a) relating the band edge energy eigenvalues and eigenfunctions at two values m and $1 - m$ of the modulus parameter. These

results are special cases of a more general treatment involving duality transformations which map the spectrum of one quasi-exactly solvable potential to that of another quasi-exactly solvable periodic potential [19].

We start from the Schrödinger equation (12). On using the relation [5] (see footnote 3)

$$\sqrt{m}\,\text{sn}(x,m) = -\text{dn}[ix + K'(m) + iK(m), 1 - m] \tag{13}$$

and defining a new variable $y = ix + K'(m) + iK(m)$, the Schrödinger equation (12) takes the form

$$-\psi''(y) + a(a+1)(1-m)\text{sn}^2(y, 1-m)\psi(y) = [a(a+1) - E]\psi(y). \tag{14}$$

On comparing equations (12) and (14), we then have the desired relations relating the eigenstates when the modulus parameter is m and $1 - m$:

$$E_j(m) = a(a+1) - E_{2a-j}(1-m) \qquad \psi_j(x,m) \propto \psi_{2a-j}(y, 1-m) \qquad j = 0, 1, \ldots, 2a. \tag{15}$$

Thus, for any integer a, at $m = 1/2$, one has remarkable relations

$$E_j(m = 1/2) + E_{2a-j}(m = 1/2) = a(a+1) \qquad E_a(m = 1/2) = a(a+1)/2 \tag{16}$$

$$\psi_j(x, m = 1/2) \propto \psi_{2a-j}(y, m = 1/2).$$

On using

$$\text{dn}(x,m) = \sqrt{1-m}\,\text{sn}[ix + K'(m) + iK(m), 1 - m]$$
$$\sqrt{m}\,\text{cn}(x,m) = i\sqrt{1-m}\,\text{cn}[ix + K'(m) + iK(m), 1 - m] \tag{17}$$

one can immediately and explicitly verify the relations (15) and (16) for $a = 1, 2, 3, 4$.

For the $a = 2$ case, the Lamé potential has two bound bands and a continuum band. The energies and wavefunctions of the five band edges are well known [2, 3]. The lowest energy band ranges from $2 + 2m - 2\delta$ to $1 + m$, the second energy band ranges from $1 + 4m$ to $4 + m$ and the continuum starts at energy $2 + 2m + 2\delta$, where $\delta = \sqrt{1 - m + m^2}$. For subsequent application of the supersymmetric quantum mechanics formalism, it is convenient to have a potential $V_-(x)$ whose ground state energy is zero. This is easily accomplished via a re-defined potential with the ground state energy subtracted out. For the problem under consideration, one has $V_-(x) = 6m\,\text{sn}^2(x,m) - 2 - 2m + 2\delta$. This potential $V_-(x)$ is plotted in figure 1. The eigenstates of all the band edges are given in table 1. Note that in the interval $2K(m)$ corresponding to the period of the Lamé potential, the number of nodes increases with energy, in agreement with the oscillation theorem. From table 1, it is easy to check that, as expected, the relations (15) and (16) are indeed true.

We might add here that in the case that a is not an integer, then for Lamé potentials one has an infinite number of bands and band gaps for which (to the best of our knowledge) no analytic results are available. However, when a is half-integer, then $(2a+1)/2$ mid-band levels are analytically known and each of them is doubly degenerate [2, 3].

3.2. Associated Lamé potentials

We now expand our discussion to the band edges and wavefunctions of a much richer class of periodic potentials given by equation (2) called AL potentials, since the corresponding Schrödinger equation is called the associated Lamé equation [2]. More precisely, we often refer to the AL potential of equation (2) as the (p, q) potential where $p = a(a+1)$ and $q = b(b+1)$ and note that $(p, 0)$ potentials are only the ordinary Lamé potentials. The AL potentials (2) can also be re-written in the alternative form $V(x) = pm\,\text{sn}^2(x) + qm\,\text{sn}^2(x + K(m))$ [5, 6]
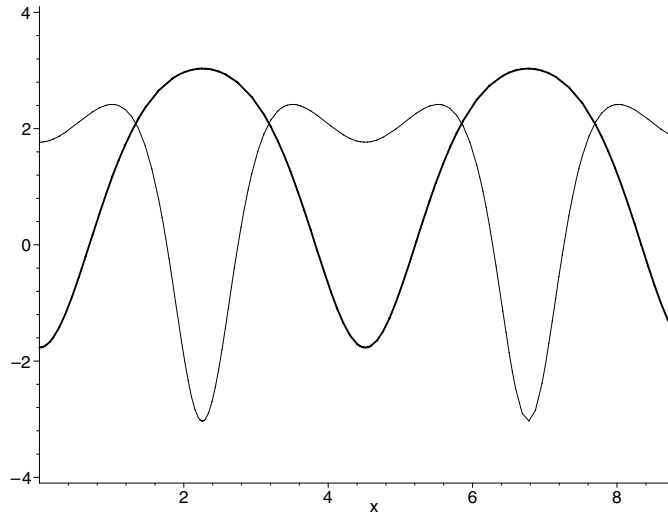
Figure 1. The $(6, 0)$ Lamé potential $V_-(x) = 6m \operatorname{sn}^2(x, m) - 2 - 2m + 2\sqrt{1 - m + m^2}$ (thick line) and its supersymmetric partner potential $V_+(x)$ (thin line) as given by equation (24). The curves are plotted for the choice $m = 0.8$. Both $V_-(x)$ and $V_+(x)$ have their lowest band edge at zero energy.

Table 1. The eigenvalues and eigenfunctions for the five band edges corresponding to the $a = 2$ Lamé potential V_- as given by equation (22) which gives $(p, q) = (6, 0)$ and its SUSY partner V_+. Here $B \equiv 1 + m + \delta$ and $\delta \equiv \sqrt{1 - m + m^2}$. The potentials $V_\pm$ have period $2K(m)$ and their analytic forms are given by equations (22) and (24), respectively. The periods of various eigenfunctions and the number of nodes in the interval $2K(m)$ are tabulated.

E	$\psi^{(-)}$	$[B - 3m \operatorname{sn}^2(x)]\psi^{(+)}$	Period	Nodes
0	$m + 1 + \delta - 3m \operatorname{sn}^2(x)$	1	$2K$	0
$2\delta - 1 - m$	$\operatorname{cn}(x)\operatorname{dn}(x)$	$\operatorname{sn}(x)[6m - (m + 1)B +$	$4K$	1
		$m \operatorname{sn}^2(x)(2B - 3 - 3m)]$		
$2\delta - 1 + 2m$	$\operatorname{sn}(x)\operatorname{dn}(x)$	$\operatorname{cn}(x)[B + m \operatorname{sn}^2(x)(3 - 2B)]$	$4K$	1
$2\delta + 2 - m$	$\operatorname{sn}(x)\operatorname{cn}(x)$	$\operatorname{dn}(x)[B + \operatorname{sn}^2(x)(3m - 2B)]$	$2K$	2
4δ	$m + 1 - \delta - 3m \operatorname{sn}^2(x)$	$\operatorname{sn}(x)\operatorname{cn}(x)\operatorname{dn}(x)$	$2K$	2

(see footnote 3). Clearly, the potentials (p, q) and (q, p) have the same energy spectra with wavefunctions shifted by $K(m)$. Therefore, it is sufficient to restrict our attention to $p \geqslant q$, i.e. $a \geqslant b$.

In general, for any values of p and q, the AL potentials have a period $2K(m)$, but for the special case $p = q$, the period is $K(m)$. From a physical viewpoint, if one thinks of a Lamé potential $(p, 0)$ as due to a one-dimensional regular array of atoms with spacing $2K(m)$ and 'strength' p, then the AL potential (p, q) results from two alternating types of atoms spaced by $K(m)$ with 'strengths' p and q respectively. If the two types of atoms are identical (which makes $p = q$), one expects a potential of period $K(m)$.

Extrema (defined for this discussion as either local or global maxima and minima) of AL potentials are easily found by setting $\mathrm{d}V(x)/\mathrm{d}x = 0$. Extrema occur when $\operatorname{sn}(x) = 0$ or
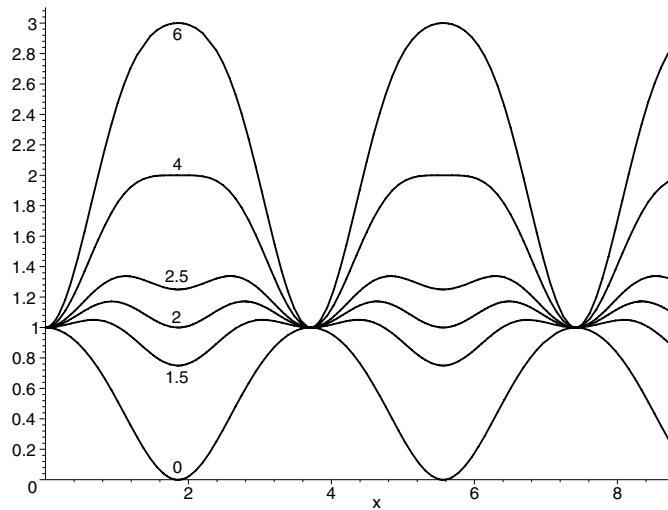
Figure 2. Plots of the (p, q) associated Lamé potentials for $q = 2, m = 0.5$ and several values of p.

$\mathrm{cn}(x) = 0$. Also, for fixed values of q and m, there are additional extrema if p lies in the critical range

$$q(1 - m) \leqslant p \leqslant q/(1 - m).$$

The AL potentials for $q = 2, m = 0.5$ and several values of p are plotted in figure 2. In the critical range of p values $1 \leqslant p \leqslant 4$, one expects additional extrema, and these are clearly seen.

3.3. Parabolas of solvability

The AL equation is

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \left[a(a + 1)m \, \mathrm{sn}^2(x) + b(b + 1)m \, \frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - E\right]\psi = 0. \tag{18}$$

On substituting $\psi(x) = [\mathrm{dn}(x)]^{-b}y(x)$, it is easily shown that $y(x)$ satisfies the Hermite elliptic equation [2]. On further substituting $\mathrm{sn}(x) = \sin t, \, y(x) \equiv z(t)$, one obtains Ince's equation

$$(1 + A \cos 2t)z''(t) + B \sin 2t \, z'(t) + (C + D \cos 2t)z(t) = 0 \tag{19}$$

where

$$A = \frac{m}{2 - m} \qquad B = \frac{(2b - 1)m}{2 - m} \qquad C = \frac{\lambda - (a + b)(a + 1 - b)m}{2 - m}$$
$$D = \frac{(a + b)(a + 1 - b)m}{2 - m} \qquad \lambda = E - mb^2 \tag{20}$$

which is a well-known quasi-exactly solvable equation [2]. In particular if $a + b + 1 = n \; (n = 1, 2, 3, \ldots)$ then one obtains n solutions, which are given in table 2. In particular, for any given choice of $p = a(a + 1)$, table 2 lists the eigenstates of the AL equation for various values of $q = b(b + 1)$.

For $q = a(a - 1)$, there is just one eigenstate with energy ma^2 and wavefunction $\psi = \mathrm{dn}^a(x)$. Since the wavefunction has period $2K(m)$ and is nodeless, this is clearly
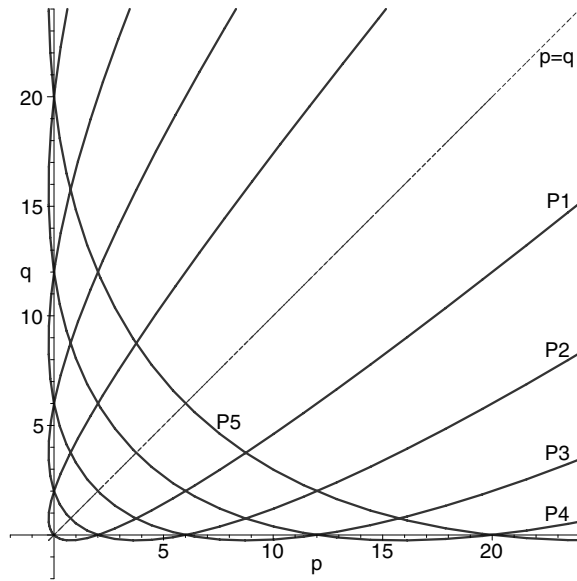
Figure 3. Parabolas of solvability. This figure illustrates all associated Lamé potentials (p, q) which are quasi-solvable. Each parabola corresponds to a choice of q in table 2. Parabola Pn is for $q = (a - n + 1)(a - n)$ for $n = 1, 2, 3, \ldots$. One knows n eigenstates for any point on parabola Pn from table 2.

Table 2. Eigenvalues and eigenfunctions for various associated Lamé potentials (p, q) with $p = a(a + 1)$ and $q = (a - n + 1)(a - n)$ for $n = 1, 2, 3, \ldots$. The periods of various eigenfunctions and the number of nodes in the interval $2K(m)$ are tabulated. Here $\delta_4 \equiv \sqrt{1 - m + m^2(a - 1)^2}$, $\delta_5 \equiv \sqrt{4 - 7m + 2ma + m^2(a - 2)^2}$, $\delta_6 \equiv \sqrt{4 - m - 2ma + m^2(a - 1)^2}$ and $\delta_7 \equiv \sqrt{9 - 9m + m^2(a - 2)^2}$.

q	E	$\mathrm{dn}^{-a}(x)\psi$	Period	Nodes
$a(a - 1)$	ma^2	1	$2K$	0
$(a - 1)(a - 2)$	$1 + m(a - 1)^2$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a - 1)(a - 2)$	$1 + ma^2$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}$	$4K$	1
$(a - 2)(a - 3)$	$2 + m(a^2 - 2a + 2) \pm 2\delta_4$	$\frac{[m(2a-1)\mathrm{sn}^2(x)-1+m-ma\pm\delta_4]}{\mathrm{dn}^2(x)}$	$2K$	$2, 0$
$(a - 2)(a - 3)$	$4 + m(a - 1)^2$	$\frac{\mathrm{sn}(x)\mathrm{cn}(x)}{\mathrm{dn}^2(x)}$	$2K$	2
$(a - 3)(a - 4)$	$5 + m(a^2 - 4a + 5) \pm 2\delta_5$	$\frac{\mathrm{cn}(x)[m(2a-1)\mathrm{sn}^2(x)-2+2m-ma\pm\delta_5]}{\mathrm{dn}^3(x)}$	$4K$	$3, 1$
$(a - 3)(a - 4)$	$5 + m(a^2 - 2a + 2) \pm 2\delta_6$	$\frac{\mathrm{sn}(x)[m(2a-1)\mathrm{sn}^2(x)-2+m-ma\pm\delta_6]}{\mathrm{dn}^3(x)}$	$4K$	$3, 1$
$(a - 4)(a - 5)$	$10 + m(a^2 - 4a + 5) \pm 2\delta_7$	$\frac{\mathrm{sn}(x)\mathrm{cn}(x)[m(2a-1)\mathrm{sn}^2(x)-3+2m-ma\pm\delta_7]}{\mathrm{dn}^4(x)}$	$2K$	$4, 2$

the ground state wavefunction of the $(a(a + 1), a(a - 1))$ potential for any real choice of the parameter a. The equations $p = a(a + 1)$ and $q = a(a - 1)$ are the parametric forms of the equation of the parabola $(p - q)^2 = 2(p + q)$, which is plotted in figure 3 and labelled $P1$. For any point on the parabola, one knows the ground state wavefunction and energy $E_0 = ma^2$. The parabola $P1$ includes the points $(2, 0)$ and $(6, 2)$.

For $q = (a-1)(a-2)$, we see from table 2 that two eigenstates with energies $1 + m(a-1)^2$ and $1 + ma^2$ are known. Since they have period $4K(m)$ and just one node in the interval $L = 2K(m)$, they must correspond to the first and second band edge energies E_1 and E_2 of the $(a(a+1), (a-1)(a-2))$ potential. Eliminating a from the equations $p = a(a+1)$ and $q = (a-1)(a-2)$ gives the 'parabola of solvability' $(p-q)^2 = 8(p+q) - 12$, which is plotted in figure 3 and labelled $P2$. This parabola includes the points $(2, 0)$ and $(6, 0)$ which correspond to Lamé potentials. Similarly, the parabolas of solvability Pn $(n = 0, 1, 2, \ldots)$ corresponding to $q = (a - n + 1)(a - n)$ in table 2 are plotted. Note that n eigenstates are known for any point on the parabola of solvability Pn.

All Lamé as well as AL potentials for which a, b are unequal integers have two parabolas of solvability passing through. This provides a good understanding of why these are completely solvable problems. For instance, the $(2, 0)$ potential is at the intersection of parabolas P1 (one known state) and P2 (two known states), thus giving the three known band edges. Similarly, the $(6, 2)$ AL potential lies on parabola P1 (1 known nodeless state of period $2K(m)$) and parabola P4 (four known states of period $4K(m)$, two with one node and two with three nodes). Since we know from the oscillation theorem that two states of period $2K(m)$ are missing, it would appear that this is an example of a quasi-exactly solvable potential. However, using a well-known theorem about Ince's equation [2] we discuss below that the two 'missing' states are degenerate and have no band gap between them. Hence, AL potentials are also exactly solvable periodic problems with a finite number of band gaps when both a and b are unequal positive integers.

Other fully solvable examples correspond to AL potentials with $a = b = $ integer. For example, the $(2, 2)$ potential has period $K(m)$. It lies on parabola P3 (three known states) and the band edge periods are $K(m), 2K(m), 2K(m)$. We shall discuss these examples in some detail in section 5 where we show that in view of the Landen transformation formulae for Jacobi elliptic functions, these problems are essentially related to Lamé potentials with integer a and hence exactly solvable.

3.4. Exact results

Several exact results are known in the literature about Ince's equation (19). In particular, it is well known that the system satisfying Ince's equation (19) has *at most* $j + 1$ band gaps of period $\pi[2\pi]$ in the case that the polynomial $Q(\mu)[Q^*(\mu)]$ defined by

$$Q(\mu) = 2A\mu^2 - B\mu - D/2 \qquad Q^*(\mu) = 2A(\mu - 1/2)^2 - B(\mu - 1/2) - D/2 \qquad (21)$$

where A, B, D are given by equation (20), has either nonnegative integral roots, the highest of which is j, or negative integral roots, the lowest of which is $-j - 1$. The fact that the Lamé potentials (1) have only a band gaps when a is any positive integer is easily understood from here. We now show [7] that on applying this theorem, one can draw the following conclusions about the associated Lamé potentials:

1. *a, b unequal positive integers.* In this case one can show that for $a > b$, there are only a bound bands followed by a continuum band out of which the lowest $a - b$ bands are normal bands with one band edge wavefunction of period $2K(m)$ and the other with period $4K(m)$, while the top b bound bands are unusual in that both their band edges are of period $2K(m)$ $(4K(m))$ in the case $a - b$ is an even (odd) integer. Further, all the $2a + 1$ band edge eigenvalues and eigenfunctions are known analytically. As an illustration, in table 3 we have given all five band edge energy eigenstates of the $(6, 2)$ potential.
2. *Both a, b being half-integral.* In this case there is an infinite number of bands and band gaps. However, except for the lowest $a - b$ bands, the rest are unusual in that both their

Table 3. The five eigenvalues and eigenfunctions for the self-isospectral associated Lamé potential corresponding to $a = 2, b = 1$ which gives $(p, q) = (6, 2)$. The potential is $V_-(x) = 6m\,\mathrm{sn}^2(x) + 2m\frac{\mathrm{cn}^2(x)}{\mathrm{dn}^2(x)} - 4m$, and has period $2K(m)$. The number of nodes in the interval $2K(m)$ is tabulated.

E	$\psi^{(-)}$	Period	Nodes
0	$\mathrm{dn}^2(x)$	$2K$	0
$5 - 3m - 2\sqrt{4 - 3m}$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}[3m\,\mathrm{sn}^2(x) - 2 - \sqrt{4 - 3m}]$	$4K$	1
$5 - 2m - 2\sqrt{4 - 5m + m^2}$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}[3m\,\mathrm{sn}^2(x) - 2 - m - \sqrt{4 - 5m + m^2}]$	$4K$	1
$5 - 2m + 2\sqrt{4 - 5m + m^2}$	$\frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}[3m\,\mathrm{sn}^2(x) - 2 - m + \sqrt{4 - 5m + m^2}]$	$4K$	3
$5 - 3m + 2\sqrt{4 - 3m}$	$\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}[3m\,\mathrm{sn}^2(x) - 2 + \sqrt{4 - 3m}]$	$4K$	3

band edges are of period $2K(m)$ $(4K(m))$ in the case $a - b$ is an even (odd) integer. Further, in this case $a - b$ band edge energy eigenstates of period $2K(m)$ $(4K(m))$ are analytically known, when $a - b$ is an odd (even) integer. Besides, one also analytically knows the energy eigenvalues and eigenfunctions of $b + 1/2$ mid-band states of period $2K(m)$ $(4K(m))$ each of which is doubly degenerate in the case $a - b$ is an odd (even) integer [7].

3. *Either $a + b$ or $a - b$ integral.* When neither a nor b is integral or half-integral but their sum or difference is an integer, then one again has an infinite number of bands and band gaps. Further, one can show that if either $a + b = 2N$ or $a - b = 2N + 1$, then there are at most $N + 1$ band gaps of period $2K(m)$ and one analytically knows the energy eigenvalues and eigenfunctions of $2N + 1$ band edges of period $2K(m)$. On the other hand, if $a + b = 2N + 1$ or $a - b = 2N$, then there are at most N band gaps of period $4K(m)$ and one has analytical expressions for $2N$ band edges of period $4K(m)$.

Finally, just as $a + 1/2$ mid-band states, with each being doubly degenerate, are analytically known in the case that parameter a of the Lamé potential (1) is half-integral, one can show that a similar number of doubly degenerate mid-band states are also known for every integral value of b in the case of AL potentials (2) with half-integral a [7].

4. Supersymmetric partner potentials

4.1. Lamé potentials

Let us first apply the supersymmetric quantum mechanics formalism as explained in section 2 to the Lamé potentials (1) when a is positive integer. Since analytic solutions are known for integer values of a [3], the supersymmetric partner potentials can be readily computed using equations (4) and (5). We first discuss the results for small integer values of a, and then present some eigenstate results for arbitrary integer values of a.

In order to use the supersymmetry formalism, we must shift the Lamé potential by a constant to ensure that the ground state (the lower edge of the lowest band) has energy $E = 0$. For $a = 1$, one has $V_-(x) = 2m\,\mathrm{sn}^2(x) - m$, $\psi_0^{(-)} = \mathrm{dn}(x)$ and the superpotential is $W = m\,\mathrm{sn}(x)\mathrm{cn}(x)/\mathrm{dn}(x)$. The partner $V_+(x)$ turns out to be just $V_-(x - K(m))$, so that this is an example of self-isospectrality. For $a = 2$, the potential is

$$V_-(x) = 6m\,\mathrm{sn}^2(x) - 2 - 2m + 2\delta \qquad \delta \equiv \sqrt{1 - m + m^2} \tag{22}$$

with a unnormalized ground state wavefunction $\psi_0^{(-)}(x) = 1 + m + \delta - 3m\,\text{sn}^2(x)$ [3]. The corresponding superpotential is

$$W = -6m\,\text{sn}(x)\text{cn}(x)\,\text{dn}(x)\big/\psi_0^{(-)}(x) \tag{23}$$

and hence the partner potential $V_+(x)$ for the potential $V_-(x)$ given in equation (22) is

$$V_+(x) = -V_-(x) + \frac{72m^2\,\text{sn}^2(x)\text{cn}^2(x)\text{dn}^2(x)}{[1 + m + \delta - 3m\,\text{sn}^2(x)]^2}. \tag{24}$$

Although supersymmetry guarantees that the potentials $V_\pm$ are isospectral, in this example they are not self-isospectral. Therefore, $V_+(x)$ as given by equation (24) is a new periodic potential which is strictly isospectral to the potential (22) and hence it also has two bound bands and a continuum band. In figure 1, we have plotted the potentials $V_\pm(x)$ corresponding to $a = 2$ for $m = 0.8$. Using equations (10) and (11) and the known eigenstates of $V_-(x)$, we can immediately compute all the band-edge Bloch wavefunctions for $V_+(x)$. In table 1, we have given the energy eigenvalues and wavefunctions for the isospectral partner potentials $V_\pm(x)$. In summary, for integral a, Lamé potentials with $a \geqslant 2$ are not self-isospectral. They have distinct supersymmetric partner potentials even though both potentials have the same $(2a + 1)$ band edge eigenvalues.

4.2. Associated Lamé potentials

It is easily checked from table 2 that the solution corresponding to $q = a(a - 1)$ as well as one of the $q = (a - 2)(a - 3)$ solutions are nodeless and correspond to the ground state. Hence, for these cases, one can obtain the superpotential and hence the partner potential V_+. For example, consider the case of $p = a(a + 1), q = a(a - 1)$. In this case W is given by $W = am\,\text{sn}(x)\text{cn}(x)/\text{dn}(x)$, so that the corresponding partner potentials are

$$V_\pm = (a \pm 1)am\frac{\text{cn}^2(x)}{\text{dn}^2(x)} + ma(a \mp 1)\text{sn}^2(x) - ma^2. \tag{25}$$

These partner potentials are self-isospectral and so supersymmetry yields nothing new. As an illustration, the potential (6, 2) as discussed in table 3 is a self-isospectral potential and supersymmetry yields nothing new.

Let us now consider the partner potential computed from the ground state for the $p = a(a + 1), q = (a - 2)(a - 3)$ case. Here $\psi_0(x) = [m(1 - a) - 1 - \delta' + m(2a - 1)\text{sn}^2(x)] \times (\text{dn}(x))^{a-2}$, where $\delta' = \sqrt{1 - m + m^2(a - 1)^2}$. The corresponding superpotential W turns out to be

$$W = \frac{m(a - 2)\text{sn}(x)\text{cn}(x)}{\text{dn}(x)} - \frac{2m(2a - 1)\text{sn}(x)\text{cn}(x)\text{dn}(x)}{[m(1 - a) - 1 - \delta' + m(2a - 1)\text{sn}^2(x)]}. \tag{26}$$

Hence the corresponding partner potentials are

$$V_- = ma(a + 1)\text{sn}^2(x) + m(a - 3)(a - 2)\frac{\text{cn}^2(x)}{\text{dn}^2(x)} - 2 - m(a^2 - 2a + 2) + 2\delta'$$

$$V_+ = -V_- + 2W^2. \tag{27}$$

These potentials are not self-isospectral. Thus one has discovered a whole new class of periodic potentials $V_+(x)$ for which three states are analytically known no matter what a is. In particular, the energy eigenfunctions of these three states are easily obtained by taking the corresponding energy eigenstates of V_- as given in table 2 and using equations (10) and (11).

We might add here that applying second-order Darboux transformations to the Lamé potentials, Fernández *et al* [17] have obtained an interesting nonlocal effect, by which the transformed potential becomes an exact or approximately displaced copy of the original one.

5. Superposition of Lamé and associated Lamé potentials

Using the recently discovered cyclic identities [12] and Landen transformation formulae [10, 11] for Jacobi elliptic functions, we shall now further expand the list of analytically solvable periodic potential problems. In particular, we obtain the exact band edge eigenstates for potentials obtained via a certain specific type of linear superposition of Lamé as well as AL potentials.

5.1. Superposition of Lamé potentials

Consider the following linear superposition of p Lamé potentials with translated arguments:

$$V(x, m) = a(a + 1)m \sum_{j=1}^{p} \text{sn}^2(x_j, m) \tag{28}$$

where $x_j \equiv x + 2(j - 1)K(m)/p$. Note that for the special case of $p = 2$, this corresponds to the AL potential with $a = b$. On expressing $\text{sn}^2(x, m)$ in terms of $\text{dn}^2(x, m)$, the corresponding Schrödinger equation is given by

$$-\psi''(x) - \left[a(a+1) \sum_{j=1}^{p} \text{dn}^2(x_j, m) \right] \psi(x) = [E - a(a+1)p]\psi(x). \tag{29}$$

On making use of the Landen transformation formula [10, 11]

$$\sum_{j=1}^{p} \text{dn}(x_j, m) = \frac{1}{\alpha} \text{dn}\left(\frac{x}{\alpha}, \tilde{m}\right) \tag{30}$$

where

$$\frac{1}{\alpha} = \sum_{j=1}^{p} \text{dn}(2(j - 1)K(m)/p, m) \tag{31}$$

and

$$\tilde{m} = (m - 2)\alpha^2 + 2\alpha^3 \sum_{j=1}^{p} \text{dn}^3(2(j - 1)K(m)/p, m) \tag{32}$$

one can rewrite the Schrödinger equation (29) in the form

$$-\psi''(x) + \frac{a(a+1)\tilde{m}}{\alpha^2} \text{sn}^2\left(\frac{x}{\alpha}, \tilde{m}\right) \psi(x) = \left[E - a(a+1)\left(p + 2A_d - \frac{1}{\alpha^2}\right)\right]\psi(x). \tag{33}$$

Here $A_d \equiv \sum_{j<k=1}^{p} \text{dn}(x_j, m)\text{dn}(x_k, m)$. Using recently discovered cyclic identities, A_d can be shown to be [12]

$$
\begin{aligned}
A_d = \frac{p}{2}\sqrt{1 - m} + p & \sum_{j=1}^{(p-2)/2} [\text{dn}(2jK(m)/p, m) \\
& - \text{cs}(2jK(m)/p, m)Z(2jK(m)/p, m)], && p \text{ even,} \\
= p & \sum_{j=1}^{(p-1)/2} [\text{dn}(2jK(m)/p, m) - \text{cs}(2jK(m)/p, m)Z(2jK(m)/p, m)], && p \text{ odd}
\end{aligned}
$$

$$\tag{34}$$

where $cs(x, m) \equiv \frac{cn(x,m)}{sn(x,m)}$, while $Z(x, m)$ is the Jacobi zeta function [5] (see footnote 3). On making the transformation $x = \alpha y$, the Schrödinger equation (33) can be rewritten as a Lamé equation but for the modulus parameter $\tilde{m}$ and with energy $E^{(L)}(\tilde{m})$:

$$-\psi''(y) + [a(a+1)\tilde{m} \, sn^2(y, \tilde{m})]\psi(y) = E^{(L)}(\tilde{m})\psi(y). \tag{35}$$

Here the true energy eigenvalues $E(m)$ of the superposed potentials (28) are related to the eigenvalues $E^{(L)}(\tilde{m})$ of the Lamé potentials by

$$E_j(m) = \frac{E_j^{(L)}(\tilde{m})}{\alpha^2} + a(a+1)\left[p + 2A_d - \frac{1}{\alpha^2}\right] \tag{36}$$

where α and A_d are given by equations (31) and (34) respectively while $\tilde{m}$ and m are related by equation (32). Similarly, the true eigenfunctions of the superposed potentials (28) are related to those of equation (35) by

$$\psi_j(x, m) \propto \psi_j^{(L)}\left(\frac{x}{\alpha}, \tilde{m}\right). \tag{37}$$

As an illustration, for $p = 2$ (i.e. $a = b$ AL case), it is well known [10, 11] that

$$\alpha = \frac{1}{1 + \sqrt{1 - m}} \qquad A_d = \sqrt{1 - m} \qquad \tilde{m} = \frac{(1 - \sqrt{1 - m})^2}{(1 + \sqrt{1 - m})^2} \tag{38}$$

and hence the energy eigenvalues of the $a = b$ AL potentials (28) (with $p = 2$) are given in terms of those of Lamé potential (1) (but with the modulus parameter $\tilde{m}$) by

$$\alpha^2 E_j^{(AL)}(m) = E_j^{(L)}(\tilde{m}) + a(a+1)\sqrt{\tilde{m}}. \tag{39}$$

For $a = 1, 2$ we can immediately verify that the energy eigenvalues and eigenfunctions of the (2, 2) and (6, 6) potentials as calculated from here are identical (as they should be) to those calculated by us by an entirely different method where no use of either Landen transformations or the cyclic identities was made [6].

In fact we can relate the energy eigenstates of the $a = b$ AL potentials at different modulus parameters by using the relationship (15) between the corresponding Lamé energy eigenstates. For this purpose, it is best to reexpress the relation (39) entirely in terms of $\tilde{m}$ by noting that on using equation (38) we have

$$m = \frac{4\tilde{m}}{(1 + \sqrt{\tilde{m}})^2}. \tag{40}$$

Using equations (38)–(40) in equation (15), we find that the AL eigenvalues are related by (where for simplicity we have replaced $\tilde{m}$ everywhere by m)

$$\frac{(1 + \sqrt{m})^2}{4} E_j^{AL}(m_1) + \frac{(1 + \sqrt{1 - m})^2}{4} E_{2a-j}^{AL}(m_2) = a(a+1)[1 + \sqrt{m} + \sqrt{1 - m}]$$
$$j = 0, 1, \ldots, 2a \tag{41}$$

where

$$m_1 = \frac{4\sqrt{m}}{(1 + \sqrt{m})^2} \qquad m_2 = \frac{4\sqrt{1 - m}}{(1 + \sqrt{1 - m})^2}. \tag{42}$$

For $a = 1, 2$, we immediately verify that these relations are indeed true. Further, the corresponding eigenfunctions are related by

$$\psi_j(x, m_1) \propto \psi_{2a-j}(y, 1 - m) \tag{43}$$

where

$$y = \frac{2 \, ix}{1 + \sqrt{m}} + K'(m) + iK(m).$$

Similarly, for $p = 3$, it is well known that

$$\alpha = \frac{1}{1 + 2q} \qquad\qquad A_d = q(q + 2)$$

$$\tilde{m} = m \frac{(1 - q)^2}{(1 + q)^2(1 + 2q)^2} \qquad q \equiv \mathrm{dn}(2K(m)/3, m) \tag{44}$$

and hence the energy eigenvalues of (28) (with $p = 3$) are given in terms of those of Lamé's with parameter $\tilde{m}$ by

$$E(m) = (1 + 2q)^2 \tilde{E}(\tilde{m}) + 2a(a + 1)(1 - q^2). \tag{45}$$

Thus, by using Landen transformations and recently discovered cyclic identities one has discovered new solvable periodic potentials. Following the last section, it is easy to see that for any integer $a \geqslant 2$, corresponding to every superposed potential, supersymmetry will give us another new exactly solvable periodic potential.

5.2. Superposition of associated Lamé potentials

Consider the following superposition of AL potentials

$$V(x, m) = a(a + 1)m \sum_{j=1}^{p} \mathrm{sn}^2(x_j, m) + b(b + 1)m \sum_{j=1}^{p} \mathrm{sn}^2(x_j + K(m), m) \tag{46}$$

where $x_j \equiv x + 2(j - 1)K(m)/p$. Now proceeding exactly as above and using the Landen transformation (30) and the cyclic identity (34), one can show that the energy eigenvalues and eigenfunctions of the superposed AL potentials (46) are related to those of AL potential (2) (but with modulus parameter $\tilde{m}$) by

$$E(m) = \frac{\tilde{E}(\tilde{m})}{\alpha^2} + [a(a + 1) + b(b + 1)] \left[p + 2A_d - \frac{1}{\alpha^2} \right] \tag{47}$$

$$\psi(x, m) \propto \psi\left(\frac{x}{\alpha}, \tilde{m}\right) \tag{48}$$

where $\alpha, \tilde{m}$ and A_d are given by equations (31), (32) and (35), respectively. Thus we have further enlarged the list of new exactly solvable periodic problems. Needless to say, using supersymmetry, one can further enlarge this list by obtaining partner potentials of the superposed potentials which will have the same band spectra.

As an illustration, for $p = 2, 3$, the energy eigenvalues of the superposed potentials (46) are given below in terms of the corresponding AL potentials (with modulus parameter m changed to $\tilde{m}$):

$$E(m) = [1 + \sqrt{1 - m}]^2 \tilde{E}(\tilde{m}) + m[a(a + 1) + b(b + 1)] \tag{49}$$

$$E(m) = (1 + 2q)^2 \tilde{E}(\tilde{m}) + 2[a(a + 1) + b(b + 1)](1 - q^2). \tag{50}$$

where use has been made of equations (38) and (44).

6. PT invariant potentials with real band spectra

We shall now show that the complex PT-invariant periodic potentials obtained by applying the anti-isospectral transformation [20] $x \rightarrow ix + \beta$ (where β is a nonzero real number) to the Lamé and AL potentials with a finite number of band gaps are also exactly solvable and

have a finite number of band gaps. We give explicit expressions for the band edges of some of these potentials [9]. For the special case of the potential $V(x) = -2m\,\text{sn}^2(ix + \beta, m)$, we analytically obtain the corresponding dispersion relation.

Let us begin with the simple observation that if $\psi(x)$ is a solution of the Schrödinger equation for the real potential $V(x)$ with energy E, then $\psi(ix + \beta)$ is a solution of the Schrödinger equation for the complex potential $-V(ix + \beta)$ with energy $-E$. The new potential $-V(ix + \beta)$, generated by the anti-isospectral transformation $x \rightarrow ix + \beta$, is clearly PT-symmetric and will be denoted by $V^{(\text{PT})}(x)$.

It is important to understand that the key point in obtaining these results is that unlike trigonometric and most other periodic functions, the Jacobi elliptic functions are doubly periodic functions. This allows both $V(x)$ and $V^{(\text{PT})}(x)$ to be simultaneously periodic, even though the periods are different. Note that the arbitrary nonzero constant β in the transformation $x \rightarrow ix + \beta$ helps us avoid the singularities of Jacobi elliptic functions.

Let us first apply our approach to the Lamé potentials (1). On applying the anti-isospectral transformation $x \rightarrow ix + \beta$ [20], where β is any nonzero real number, it is easily shown that the band-edge eigenvalues and eigenfunctions of the PT-invariant potentials

$$V^{(\text{PT})}(x) = -a(a + 1)m\,\text{sn}^2(ix + \beta, m) \qquad a = 1, 2, 3, \ldots \qquad (51)$$

are related to those of the Lamé potentials (1) by [9]

$$E_j^{(\text{PT})}(m) = -E_{2a-j}(m) \qquad \psi_j^{(\text{PT})}(x, m) \propto \psi_{2a-j}(ix + \beta, m) \qquad j = 0, 1, \ldots, 2a. \quad (52)$$

It may be noted here that the PT-invariant complex potential (51) is a periodic potential with period $2K'(m)(\equiv 2K(1 - m))$. Thus we have shown that the PT-invariant periodic potential (51) also has precisely a band gaps and hence $2a + 1$ band edges at energies given by relation (52). Special mention may be made of the remarkable fact that for any integer a, all bands and band gaps exchange their role as one goes from a Lamé potential to its PT-transformed version.

For any band structure problem, one important quantity is the discriminant Δ [2] which gives information about the number of band gaps as well as their widths. The question is whether we can relate $\Delta^{(\text{PT})}$ with the discriminant Δ for the corresponding Lamé potential. We now show that using equations (15) and (52) this is indeed possible. In particular, on using these two equations we deduce that

$$E_j^{(\text{PT})}(m) = E_j(1 - m) - a(a + 1) \qquad j = 0, 1, \ldots, 2a \qquad (53)$$

and hence the corresponding discriminants are related by [9]

$$\Delta^{(\text{PT})}(E, m) = \Delta[E + a(a + 1), 1 - m]. \qquad (54)$$

As an illustration, in figure 4 we plot the real and imaginary parts of the PT-invariant complex potential $-6m\,\text{sn}^2(ix + \beta, m)$. Using the well-known results for the Lamé potential with $a = 2$ (see table 2) and equation (52), the ground state (lowest band edge) eigenvalue and eigenfunction are easily shown to be

$$\psi_g(x) = \text{sn}(ix + \beta, m)[1 + m - \delta - 3m\,\text{sn}^2(ix + \beta, m)] \qquad E_g = -2 - 2m - 2\delta \qquad (55)$$

where $\delta = \sqrt{1 - m + m^2}$. In table 4, we have given all five band edge eigenvalues and eigenfunctions of this PT-invariant potential, where we have subtracted the ground state energy from the potential so that the lowest band edge by construction is at zero energy. Observe from the table that the band edges are both periodic as well as anti-periodic with periods of $2K'(m)$ and $4K'(m)$, respectively, thereby showing that contrary to the suggestion of Bender *et al* [21], the absence of anti-periodic band edges is not a general property of PT-invariant periodic potentials.
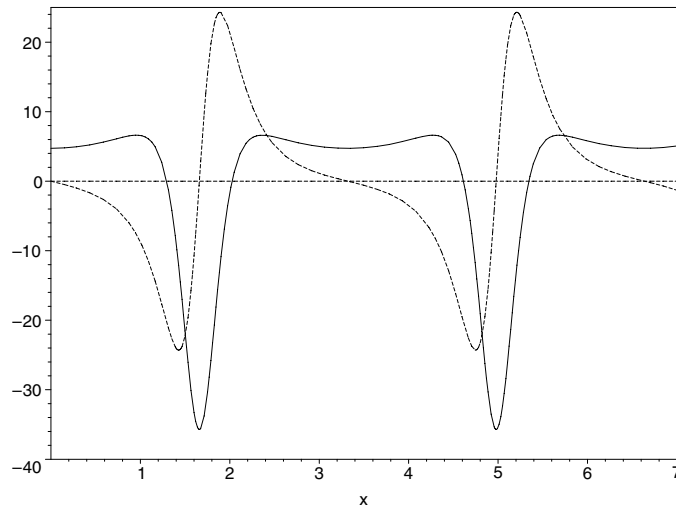
Figure 4. A plot of the real (solid line) and imaginary (dashed line) parts of the PT-invariant complex potential (51) for $a = 2$. More explicitly, the potential is $-6m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m) + 2 + 2m + 2\sqrt{1 - m + m^2}$ and has its lowest band edge at zero energy. The plot is made for the choice $m = 0.8$ and $\beta = 0.4$. The potential has a period $2K'(0.8) = 3.3192$.

Table 4. The eigenvalues and eigenfunctions for the five band edges corresponding to the PT-invariant potential $V_-^{(\mathrm{PT})} = -6m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m) - E_g$ where $E_g = -2 - 2m - 2\sqrt{1 - m + m^2}$ (equation (55)). The potential has period $2K'(m)$. The periods of various eigenfunctions are also tabulated.

E	$\psi^{(-)}$	Period
0	$1 + m - \delta - 3m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m)$	$2K'(m)$
$m - 2 + 2\delta$	$\mathrm{cn}(\mathrm{i}x + \beta, m)\mathrm{sn}(\mathrm{i}x + \beta, m)$	$4K'(m)$
$1 - 2m + 2\delta$	$\mathrm{sn}(\mathrm{i}x + \beta, m)\mathrm{dn}(\mathrm{i}x + \beta, m)$	$4K'(m)$
$1 + m + 2\delta$	$\mathrm{cn}(\mathrm{i}x + \beta, m)\mathrm{dn}(\mathrm{i}x + \beta, m)$	$2K'(m)$
4δ	$1 + m + \delta - 3m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m)$	$2K'(m)$

Additional analytically solvable finite band gap potentials can be obtained from here by using supersymmetry. In particular, consider the potential $V_-^{(\mathrm{PT})}(x) = -6m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m) - E_g$. Its ground state eigenfunction is given in equation (55), and we find that the corresponding superpotential is

$$W^{(\mathrm{PT})}(x) = -\mathrm{i}\frac{\mathrm{cn}(\mathrm{i}x + \beta, m)\mathrm{dn}(\mathrm{i}x + \beta, m)}{\mathrm{sn}(\mathrm{i}x + \beta, m)} + 6\,\mathrm{i}m\frac{\mathrm{cn}(\mathrm{i}x + \beta, m)\mathrm{sn}(\mathrm{i}x + \beta, m)\mathrm{dn}(\mathrm{i}x + \beta, m)}{[1 + m - \delta - 3m\,\mathrm{sn}^2(\mathrm{i}x + \beta, m)]}.$$
(56)

Thus the supersymmetric partner potential is

$$V_+^{(\mathrm{PT})}(x) = [W^{(\mathrm{PT})}]^2(x) + [W^{(\mathrm{PT})}]'(x)$$
(57)

and it has the same energy spectrum as $V_-^{(\mathrm{PT})}(x)$ (see table 4). In this way, one has discovered yet another PT-invariant complex potential with a finite number of band gaps. The corresponding band edge energy eigenfunctions are immediately obtained from those of $V_-^{(\mathrm{PT})}$ by using supersymmetry transformations given in section 2.

Yet another complex PT-invariant potential having the same band edge eigenvalues as above is obtained by starting from the partner potential $V_+(x)$ of the $a = 2$ Lamé potential (equations (24) and (22)) and then obtaining the corresponding PT-invariant potential by the anti-isospectral transformation $x \to ix + \beta$. Thus one has found three distinct complex PT-invariant periodic potentials having the same band spectra. One can show that this is true for any integral $a \geqslant 2$. On the other hand, for $a = 1$ one can show [9] that the three potentials are not distinct but are self-isospectral (i.e. differ from each other by a translation by a constant) [14, 17].

For the special case of the $a = 1$ Lamé potential, the dispersion relation is analytically known [4]. We now show [9] that it is possible to derive the dispersion relation for the PT-invariant potential (51) with $a = 1$. To that end we start from the Schrödinger equation:

$$-\psi''(x) + [1 + m - 2m \, \mathrm{sn}^2(ix + \beta, m)]\psi(x) = E\psi(x) \tag{58}$$

where we have subtracted the ground state energy from the potential so that the new potential has zero ground state energy. On substituting $y = ix + \beta$, equation (58) takes the form

$$-\psi''(y) + [2m \, \mathrm{sn}^2(y, m) - m]\psi(y) = (1 - E)\psi(y). \tag{59}$$

Now it is well known that two independent solutions of this equation are given by [4]

$$\psi(x) = \frac{H(ix + \beta \pm \alpha_1) \exp[\mp(ix + \beta)Z(\alpha_1)]}{\theta(ix + \beta)} \tag{60}$$

where H, θ, Z are Jacobi eta, theta and zeta functions while α_1 is related to E of equation (58) by

$$E = m \, \mathrm{sn}^2(\alpha_1, m). \tag{61}$$

On using the fact that while $\theta(ix + \beta)$ is a periodic function with period $2K'(m)$, $H(ix + \beta)$ is only quasi-periodic [4], i.e.

$$H(i[x + 2K'(m)] + \beta) = H(ix + \beta) \exp[-\pi K'(m)/K(m)] \tag{62}$$

and using the Bloch condition, it is easily shown that for the PT-invariant complex potential (1) with $a = 1$, the dispersion relation is given by

$$k = \mp \frac{\pi}{2K'(m)} \pm iZ(\alpha_1) + i\frac{\pi}{2K(m)} \tag{63}$$

where α_1 is given by equation (61).

We now turn to the associated Lamé potentials given by equation (2) where without any loss of generality we consider $a > b$ with both being positive integers. On using the anti-isospectral transformation, it is easy to see that the band edges of the potential (2) and its PT-transformed one as given by

$$V(x) = -a(a + 1)m \, \mathrm{sn}^2(ix + \beta, m) - b(b + 1)m\frac{\mathrm{cn}^2(ix + \beta, m)}{\mathrm{dn}^2(ix + \beta, m)} \tag{64}$$

are again related by the relation (52).

For the special case $a = b$, using relations (40), (41) and (52) we obtain an interesting relation between the eigenvalues of the AL potential and the corresponding complex PT-invariant potential given by

$$E_j^{(\mathrm{PT})}(m_1) = \left(\frac{1 + \sqrt{1 - m}}{1 + \sqrt{m}}\right)^2 E_j(m_2) - 4a(a + 1)\frac{(1 + \sqrt{m} + \sqrt{1 - m})}{(1 + \sqrt{m})^2} \tag{65}$$

where $m_{1,2}$ are as defined by equation (42). Hence we can immediately relate the discriminant Δ for the $a = b$ AL potential and the corresponding complex PT-invariant potential, i.e. we have

$$\Delta^{(PT)}(E, m_1) = \Delta\left[\left(\frac{1 + \sqrt{m}}{1 + \sqrt{1-m}}\right)^2 \left(E + \frac{4a(a+1)(1 + \sqrt{m} + \sqrt{1-m})}{(1 + \sqrt{m})^2}, m_2\right)\right]. \quad (66)$$

7. Unusual band structure—the double sine-Gordon equation

Finally, we would like to consider the band spectra of the double sine-Gordon equation characterized by the periodic potential

$$V(x) = b^2 \sin^2 2x + 2ab \cos 2x \quad (67)$$

where a, b are real, and without loss of generality a is taken to be nonnegative. This potential, sometimes called the trigonometric Razavy potential, arises in several areas of condensed matter physics. It is well known that this system has an infinite number of bands and band gaps. Further, it is also well known that if a is an integer then a band edges of period π (2π) are analytically known depending on whether a is an odd (even) integer [22–24]. However, what is not so well known [2] is that in the case a is an integer, then the potential has unusual band spectra.

We start from the Schrödinger equation for the potential (67). On using the ansatz

$$\psi(x) = \exp(-b \cos 2x/2)\phi(x) \quad (68)$$

in the Schrödinger equation, it is easily shown that ϕ satisfies Ince's equation (19)

$$\phi''(x) + 2b \sin 2x \phi'(x) + [E - 2b(a-1)\cos 2x]\phi(x) = 0. \quad (69)$$

As discussed before, this is a quasi-exactly solvable equation, i.e. a band edges of period π (2π) are analytically known when a is an odd (even) integer [22, 23]. We can now compute the polynomials $Q(\mu)$ and $Q^*(\mu)$ defined in equation (21), and from them we conclude that if a is an odd (even) integer then there are at most $\frac{a+1}{2}$ $\left(\frac{a+2}{2}\right)$ band gaps of period π (2π).

It is worth pointing out here that similar conclusions can also be drawn about the complex PT-invariant periodic potential

$$V(x) = -b^2 \sin^2 2x + 2iab \cos 2x. \quad (70)$$

Note that this potential is invariant under the parity transformation $x \to x + \pi/2$ followed by T. As shown in [23], when a is an even integer PT symmetry is spontaneously broken and there are no quasi-exactly solvable states of period 2π since in this case all the eigenvalues are now complex conjugate pairs. However, if a is an odd integer then there are a quasi-exactly solvable states of period π and in this case PT symmetry is not spontaneously broken. On completing the above analysis, it follows that if a is an odd integer then there are at most $(a+1)/2$ band gaps of period π. Thus, in this case the band spectrum is rather unusual in that the majority of the infinite number of band edges are anti-periodic and have period 2π, thereby again showing that the absence of anti-periodic band edges is not a characteristic feature of PT-invariant potentials.

Acknowledgment

References

[1] See, for example, Cohen-Tannoudi C, Laloë F and Diu B 1973 *Mecanique Quantique* (Paris: Hermann)
[2] Magnus W and Winkler S 1966 *Hill's Equation* (New York: Wiley)
[3] Arscott F M 1981 *Periodic Differential Equations* (Oxford: Pergamon)
[4] Whittaker E T and Watson G N 1980 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[5] For the properties of Jacobi elliptic functions, see, for example, Gradshteyn I S and Ryzhik I M 2000 *Table of Integrals, Series and Products* (New York: Academic)
    Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (New York: Dover)
[6] Khare A and Sukhatme U 1999 *J. Math. Phys.* **40** 5473
[7] Khare A and Sukhatme U 2001 *J. Math. Phys.* **42** 5652
[8] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
    For a recent review of this field, see Bender C M, Brody D C and Jones H F 2003 *Am. J. Phys.* **71** 1095 and references therein
[9] Khare A and Sukhatme U 2004 *Phys. Lett.* A **324** 406
[10] See, for example, Lawden D F 1989 *Elliptic Functions and Applications (Applied Mathematical Science* vol 80*)* (Berlin: Springer)
    Cayley A 1895 *An Elementary Treatise on Elliptic Functions* (London: Bell)
[11] Khare A and Sukhatme U 2003 *Preprint* math-ph/0312074
[12] Khare A and Sukhatme U 2002 *J. Math. Phys.* **43** 3798
    Khare A, Lakshminarayan A and Sukhatme U 2003 *J. Math. Phys.* **44** 1822 (*Preprint* math-ph/0306028)
[13] See, for example, Cooper F, Khare A and Sukhatme U P 1995 *Phys. Rep.* **251** 267
[14] Dunne G and Feinberg J 1998 *Phys. Rev.* D **57** 1271
[15] Brihaye Y and Godart M 1993 *J. Math. Phys.* **34** 5283
    Brihaye Y and Braibant S 1993 *J. Math. Phys.* **34** 2107
    Braden H and Macfarlane A 1995 *J. Phys. A: Math. Gen.* **18** 3151
    Dunne G and Mannix J 1998 *Phys. Lett.* B **428** 115
[16] Fernández D J, Negro J and Nieto L M 2000 *Phys. Lett.* A **275** 338
    Fernández D J, Mielnik B, Rosas-Ortiz O and Samsonov B F 2002 *J. Phys. A: Math. Gen.* **35** 4279
[17] Fernández D J, Mielnik B, Rosas-Ortiz O and Samsonov B F 2002 *Phys. Lett.* A **294** 168
    Samsonov B F, Glasser M L, Negro J and Nieto L M 2003 *J. Phys. A: Math. Gen.* **36** 10053
[18] See, for example, Patera J and Winternitz P 1973 *J. Math. Phys.* **14** 1130
[19] Dunne G V and Shifman M 2002 *Ann. Phys., NY* **299** 143
[20] Krajewska A, Ushveridze A and Walczak Z 1997 *Mod. Phys. Lett.* A **12** 1225
[21] Bender C M, Dunne G V and Meisinger P N 1999 *Phys. Lett.* A **252** 272
[22] Razavy M 1980 *Am. J. Phys.* **48** 285
[23] Khare A and Mandal B P 1998 *J. Math. Phys.* **39** 3476
[24] Finkel F, Gonzalez-Lopez A and Rodriguez M A 1999 *J. Phys. A: Math. Gen.* **32** 6821